Equivariant K-theory and index theory

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Equivariant K-theory

Definition

Let G be a compact Lie group. We define a G-space X to be a space X equipped with a continuous action $G \curvearrowright X$. We define a G-map of G-spaces to be a map of spaces preserving the G-action.
We define a pointed G-space by a pair $(X, *)$, where $*$ is a G-fixed point of X.
For G-maps of G spaces $f, g : X \to Y$, we define f to be homotopic to g if there is a map $h : [0,1] \times X \to Y$ such that $h(t, -)$ is a G-map for all $t \in [0,1]$, and $h(0, -) = f$, $h(1, -) = g$.
Just in the previous talk on K-theory, we will mainly focus on compact G-spaces where the vector bundles over it behaves nicely.

Theorem

Let G be a compact Lie group. Then there is a unique up to scalar Haar measure μ over G such that the following condition holds.

1. For any open set U, $\mu(U) \neq 0$.
2. For any $g \in G$, and S a measureable set in G, then $\mu(S) = \mu(gS) = \mu(Sg)$.

Definition

Let X be G-space, then we define an G-vector bundle E to be a G-space with a G-map $p : E \to X$, such that the two following condition holds.

1. $p : E \to X$ is a complex vector bundle on X.
2. For any $g \in G$ and $x \in X$, the group action $g : E_x \to E_{gx}$ is a linear map of vector spaces.

Let E, F be G-vector bundles over X. We define a map of G-vector bundles $f : E \to F$ to be a G-map commuting with the structure maps of the vector bundles. Such f is an isomorphism if it admits an inverse, which is equivalent to that f is an isomorphism pointwise.

Example

Let X be a trivial G-space, then any G-representation over $\mathbb{C}$, say V, gives a G-vector bundle over X. We denote this by $\underline{V}$.
Let X be a G-space, then the tangent bundle TX is a G-vector bundle over X.

Construction

For two G-vector bundles over X, say $p : E \to X$ and $q : F \to X$, we may construct the following:

1. The equivariant sections, denoted by $\Gamma^G(X; E)$, which is a complex vector space with the elements G-maps $s : X \to E$ such that $ps = \mathrm{id}_X$. We may also forget the G-structures and form the usual section $\Gamma(X; E)$. By choosing a Hermitian metric on E, $\Gamma(X; E)$ becomes a Banach space with a continuous action of G.
2. Their direct sum, denoted by $E \oplus F$, with its underlying space the direct sum bundle $E \oplus F$ equipped with the action $g(e, f) = (ge, gf)$, where $g \in G$, $e \in E$, $f \in F$.
3. Their tensor product, denoted by $E \otimes F$, again with the underlying space $E \otimes F$ and equipped with the action $g(e \otimes f) = ge \otimes gf$.
4. Their Hom bundle, denoted by $\mathrm{Hom}(E, F)$, with again the underlying space $\mathrm{Hom}(E, F)$, and the G action given by $(gh)(e) = g \cdot h(g^{-1}e)$ $h \in \mathrm{Hom}(E, F)_x$, $e \in E_{gx}$.

Construction

Note that the Hom bundle generalizes the notion of maps of bundles. In fact, a map $f : E \to F$ of G-vector bundles over X corresponds to an element in $\Gamma^G(X; \mathrm{Hom}(E, F))$.

Moreover, for a G-map of G-spaces $f : X \to Y$, and for any G-vector bundle $E \to Y$, we may construct the pullback G-vector bundle $f^*E \to X$, with the obvious underlying vector bundle and actions. This construction clearly preserves the direct sum, tensor product, and Hom of G-vector bundles.

This pullback construction generalizes to the case where $\varphi : H \to G$ is a map of compact Lie group, and $f : X \to Y$ is a map of spaces (where X is an H-space, and Y is an G-space) such that $f(hx) = \varphi(h)f(x)$.

We have a theorem for pullback bundles.

Theorem

If $f, g : X \to Y$ are G-homotopic G-maps of compact G-spaces, and $p : E \to Y$ is a G-vector bundle over Y, we have $f^*E \simeq g^*E$ as G-vector bundles over X.

proof

We define an isomorphism $k : f^*E \to g^*E$. Explicitly, suppose the homotopy between f and g be $h : [0,1] \times X \to Y$, we let $k_x : f^*E_x \to g^*E_x$ be given by the composition

$$f^*E_x \simeq E_{f(x)} \xrightarrow{\tilde{h}} E_{g(x)} \simeq g^*E_x$$

where $\tilde{h}$ is the G-equivariant parallel transport along the curve $h(-, x)$ (this is locally given by the trivializations of the bundle). This clearly gives a well-defined G-map $f^*E \to g^*E$, and by construction, it is a pointwise isomorphism, hence is an isomorphism.

Definition

Let X be a compact G-space. Then all G-vector bundles over X form a commutative monoid over X with multiplication the direct sum of isomorphism classes of bundles. We define the equivariant K-theory of X to be $KU_G(X)$, which is the group completion of the above commutative monoid. Explicitly, its elements are virtual classes of the form $[E_1 - E_2]$, where E_1 and E_2 are vector bundles over X, and $[E_1 - E_2] \simeq [E_1' - E_2']$ iff there is some G-vector bundle F such that $E_1 \oplus E_2' \oplus F \simeq E_1' \oplus E_2 \oplus F$. $KU_G(X)$ can be given a ring structure by the tensor product of vector bundles.
As in K-theory, for a map $f : X \to Y$ of compact G-spaces, the pullback of G-vector bundles over Y along f gives a ring homomorphism $f^* : KU_G(Y) \to KU_G(X)$. Moreover, as in the generalization of the pullback, we have a map $f^* : KU_G(Y) \to KU_H(X)$.

Corollary

If $f, g : X \to Y$ are G-homotopic maps of compact G-spaces, then we have $f^* \simeq g^* : KU_G(Y) \to KU_G(X)$.

proof

This follows from the previous theorem, stating that $f^*E \simeq g^*E$ whenever f and g are G-homotopic.

Example

$KU_e(X) = KU(X)$, where e is the trivial group.
$KU_G(*) = R(G)$, where $R(G)$ is the representation ring of G. In particular, we have every $KU_G(X)$ a $R(G)$-algebra, since the unique map $X \to *$ gives a ring homomorphism $R(G) \to KU_G(X)$.
$KU_G(G/H) = R(H)$, where H is a closed subgroup of G. More generally, for a map $\varphi : G \to H$ and an H-space X, we have $KU_G(G \times_H X) \simeq KU_H(X)$. (The equivalence is given by the pullback map along $X \hookrightarrow G \times_H X$ and the extension of bundles $E \mapsto G \times_H E$).

Proposition

If N is a normal closed subgroup of G such that N acts freely on X, then we have $p^* : KU_{G/N}(X/N) \to KU_G(X)$ is an equivalence.
If X is a trivial G-space, then the natural map $R(G) \otimes KU(X) \to KU_G(X)$ is an isomorphism.

proof

For the first assertion, note $X/N \simeq (G/N) \times_G X$ since the N-action is free, and hence $KU_{G/N}(X/N) \simeq KU_G(X)$.
For the second assertion, note that since the action of G on X is trivial, for any G-vector bundle $E \to X$, we have $g : E_x \to E_x$. Note that by the Haar measure, on each E_x, we have a projection $E_x \to E_x^G$ (the invariant subspace) by integration

$$v \mapsto \int_G gv\mathrm{d}g$$

This gives a map of rings $\varepsilon : KU_G(X) \to KU(X)$. Note that this gives an isomorphism $E \simeq \bigoplus_{V irr} \underline{V} \otimes \mathrm{Hom}(\underline{V}, E)^G$ by verifying pointwise.

proof

Therefore

$$KU(X) \otimes R(G) \to KU_G(X), (E, V) \mapsto E \otimes \underline{V}$$

$$KU_G(X) \to KU(X) \otimes R(G), E \mapsto \bigoplus_{V irr} \underline{V} \otimes \mathrm{Hom}(\underline{V}, E)^G$$

gives the isomorphism.

We state the following theorem without proof (since it relies on hardcore functional analysis).

Theorem

Let E be a G-vector bundle on a compact G-space X, then there is some $V \in R(G)$, and an isomorphism $\underline{V} = E \oplus E^{\perp}$.

(This allows us to give the reduced construction of K and investigate its homological properties).

Definition

We define the reduced K-theory, $\widetilde{KU}_G(X)$ to be the set $KU_G(X)/\sim$, where $\sim$ is the relation $E \sim F$ iff $E \oplus \underline{V} \simeq F \oplus \underline{V'}$. Equivalently, we may define the reduced K-theory of a space X to be the kernel of the map $KU_G(X) \to KU_G(*)$, induced by an arbitrary base point $* \to X$. Note that this is naturally an Abelian group, but only admits a non-unital commutative ring structure.

Theorem

Let X be a pointed compact G-space, and let A be a closed G-subspace of X equipped with(and in particular containing) the same basepoint. Then we have an exact sequence.

$$\widetilde{KU}_G(X/A) \to \widetilde{KU}_G(X) \to \widetilde{KU}_G(A)$$

Here X/A is the G-space $X \amalg_A CA$, where CA is the cone of A.

proof

The sequence is obviously a chain complex. For exactness, note that if $E \in \widetilde{KU}_G(X)$ vanishes in $\widetilde{KU}_G(A)$, then there are trivial G-vector bundles over A such that $E|_A \oplus \underline{V} \simeq \underline{V'}$. Thus $E \oplus \underline{V} \in KU_G(X)$ and $\underline{V'} \in KU_G(CA)$ can be clutched together on A, which gives the desired bundle in $KU_G(X/A)$ and hence $\widetilde{KU}_G(X/A)$.

Definition

Let X be a locally compact G-space. We define X^+ to be the one-point-compactification of X, equipped with a G-action that fixes ∞ and the inclusion $X \hookrightarrow X^+$ is a G-map.
Let $q \in \mathbb{N}$. We define the following.

1. $KU_{G,c}^{-q}(X) = \widetilde{KU}_G^{-q}(X^+) = \widetilde{KU}_G(S^q X^+)$.
2. $KU_{G,c}^{-q}(X, A) = \widetilde{KU}_G(S^{-q}(X^+ \amalg_{A^+} C(A^+)))$

We sometimes write $KU_{G,c}$ for $KU_{G,c}^0$ for short. In particular, by the definition of smash products, we have a multiplication map $KU_{G,c}(X) \otimes KU_{G,c}(Y) \to KU_{G,c}(X \times Y)$. Moreover, as in the non-equivariant case, $KU_{G,c}(X)$ is again equivalent to the chain complexes of vector bundles with compact support.

As the notations indicates and the preceding theorem shows, this is the negative part of a generalized cohomology theory, and so the excision theorem follows.

The equivariant Thom isomorphism

As in the non-equivariant case, we have the Thom isomorphism.

Theorem

Let E be a G-vector bundle over a compact G-space X, then there is some $\lambda_E \in KU^0_{G,c}(E)$ such that the multiplication map $KU_{G,c}(X) \to KU_{G,c}(E)$ given by $\alpha \mapsto p^*\alpha \cdot \lambda_E$ is an isomorphism.

To prove this theorem, we need the concept of equivariant index theory.

Definition

Let V and W be Banach spaces equipped with a G-action. We define an G-operator $f : V \to W$ to be a bounded operator respecting the G-action. We define this operator f to be Fredholm if $\ker f$ and $\operatorname{coker} f$ are both finite dimensional. Then, we define $\operatorname{ind} f = \ker f - \operatorname{coker} f \in R(G)$. For a G-bundle E over a G-smooth manifold X, recall $C^\infty(X; E)$ is a Banach space equipped with a G-action. We define a G-differential operator $D : C^\infty(X; E) \to C^\infty(X; F)$ to be a differential operator commuting with the action of G. In particular, we say it is elliptic if it is elliptic as a differential operator.

We first prove the local case, which is the equivariant version of Bott periodicity.

Proposition

Let V be a finite dimensional complex G-representation, and X a compact G-space. Then there is some λ_V such that the multiplication map $\varphi = \cdot\lambda_V : KU_{G,c}(X) \to KU_{G,c}(X \times V)$ is an isomorphism.

proof

We first note that $KU_{G,c}(X) \simeq KU_G(X)$, hence it is a ring. It then suffices to show that there is some map $\psi : KU_{G,c}(X \times V) \to KU_G(X)$ which is a $KU_G(X)$-module homomorphism functorial in X, and $\psi(\lambda_V) = 1$. In particular, ψ is a left inverse for φ.
If this is the case, then for $\beta \in KU_{G,c}(X \times V)$, $\varphi\psi(\beta) = \psi(\beta) \cdot \lambda_V = \psi(\beta \cdots \lambda_V) = \psi(\beta)\lambda_V = \beta$. In fact, this only uses the functorality of ψ for the case $X = X \times V$.

proof

To construct the desired map, consider the projective space $\mathbb{P} = \mathbb{P}(V \oplus \mathbb{C})$. Then by the existence of the Haar measure, we may assume that we have a G-invariant Hermitian metric on $\mathbb{P}$.

We first construct λ_V. As in the non-equivariant case, we let λ_V denote the class in $KU_{G,c}(V)$ representing the bundle of exterior algebras, which is presented by the chain complex

$$0 \to \bigwedge^0 V \to \bigwedge^1 V \to \cdots \to \bigwedge^n V \to 0$$

We further let λ_V^i denote each component in the complex. Note that λ_V generates $KU_{G,c}(V)$.

proof

We then construct a map $j : KU_{G,c}(V) \to KU_G(\mathbb{P})$. Note that by the splitting principle, we have the following splitting of the pullback bundle

$$\begin{tikzcd} \mathcal{O}(-1) \oplus V' \arrow[r] \arrow[d] & V \oplus \mathbb{C} \arrow[d] \\ \mathbb{P} \arrow[r] & * \end{tikzcd}$$

Then the map $\mathcal{O}(-1) \hookrightarrow \underline{V \oplus \mathbb{C}} \twoheadrightarrow \underline{V}$ gives a map

$$\underline{\mathbb{C}} \to \mathcal{O}(1) \otimes \underline{V}$$

proof

This gives a construction of a chain complex

$$0 \longrightarrow \mathcal{O}(0) \longrightarrow \mathcal{O}(1) \otimes \bigwedge^1 V \longrightarrow \cdots \longrightarrow \mathcal{O}(n) \otimes \bigwedge^n V \longrightarrow 0$$

supported on the above section. This is clearly a well-defined element of $KU_G(\mathbb{P})$, and hence if we let $j(\lambda_V)$ be the above chain complex, or explicitly

$$j(\lambda_V) = \sum (-1)^i \mathcal{O}(i) \lambda_V^i$$

then we have a well-defined map $j : KU_{G,c}(V) \to KU_G(\mathbb{P})$.

proof

Then we let

$$\Omega^+ = \bigoplus \Omega^{0,2k}, \Omega^- = \bigoplus \Omega^{0,2k+1}$$

Consider the G-elliptic operator $D = \overline{\partial} + \overline{\partial}^* : \Omega^+ \to \Omega^-$. Then, for an arbitrary holomorphic vector bundle Q over $\mathbb{P}$, we may consider

$$D \otimes \mathrm{id}_Q = D_Q : \Omega^+ \otimes Q \to \Omega^- \otimes Q$$

By the Hodge decomposition theorem, we can identify the kernel and cokernel of D_Q as

$$\ker D_Q = \bigoplus H^{2k}(M; \Omega^{0,\bullet}(Q))$$

$$\operatorname{coker} D_Q = \bigoplus H^{2k+1}(M; \Omega^{0,\bullet}(Q))$$

proof

Finally we may calculate its index in $R(G)$

$$\operatorname{ind} D_Q = \bigoplus (-1)^i H^i(M; \Omega^{0,\bullet}(Q))$$

Here we identify these cohomologies with the corresponding harmonic forms. Then we calculate these cohomologies for the case $Q = j(\lambda_V)$. In fact, by Kodaira vanishing theorem, we have for $Q = j(\lambda_V)$, $D_Q = 1$ since all higher cohomology vanishes. Therefore we may define the initial splitting as the map

$$\psi : KU_{G,c}(V \times X) \xrightarrow{j} KU_{G,c}(\mathbb{P}(V \oplus \mathbb{C}) \times X) \xrightarrow{(\operatorname{ind} D)\cdot -} KU_{G,c}(X)$$

By the above constructions, we have $\psi(\lambda_V) = 1$, and the functorality and module homomorphism is direct.

Now we prove the initial Thom isomorphism.

proof

For the Thom isomorphism, let E be a G-vector bundle over a compact G-space X we let $H = G \times U(n)$, and let Y be the associated $U(n)$-principal bundle of E over X, so we have

$$KU_{G,c}(X) \simeq KU_{H,c}(Y) \simeq KU_{H,c}(V \times Y) \simeq KU_{G,c}(E)$$

The equivariant index theorem

Definition

Let $f : X \to Y$ be a smooth proper G-embedding of compact G-manifolds, with the G-tubular neighborhood N given by [Kan07]. Then note that the G-tubular neighborhood of $\mathrm{d}f : TX \to TY$ can be identified with TN. If we let the Thom isomorphism for $TN \to TX$ be $\tau_f : KU_{G,c}(TX) \to KU_{G,c}(TN)$. Then since $TN \to TY$ is open, we have a map

$$TX^+ \to TX^+/(TX^+ \backslash TN) \simeq TN^+$$

which gives a pushforward $j_* : KU_{G,c}(TN) \to KU_{G,c}(TY)$. Thus we have a proper pushforward map

$$f_! = j_* \circ \tau_f : KU_{G,c}(TX) \to KU_{G,c}(TY)$$

We first present the definition of the analytic and topological index.

Definition

Let X be a compact G-manifold, by integration via the Haar measure, we may choose a G-invariant Riemannian metric over X. Then consider the equivariant map $\pi : TX \to X$. Now, for two G-vector bundles E, F over X, consider a G-elliptic operator $P : C^\infty(E) \to C^\infty(F)$. By taking values on a covector, $\sigma(P)$ defines a map of G-vector bundles over TX

$$\sigma(P) : \pi^* E \to \pi^* F$$

Since P is elliptic, $\sigma(P)$ is supported on the zero section (as a chain complex of vector bundles). Thus P defines a class $\sigma(P) \in KU_{G,c}(TX)$. The converse of the above construction is also true, which gives a map $\mathrm{ind}_a : KU_{G,c}(TX) \to R(G)$, by assigning a elliptic operator to its analytic index.

Definition

For the topological index, note that again we need a G-equivariant embedding of X into some finite dimensional representation of G. Since X is compact, the embedding is provided by [Pal57]. Suppose the embedding is $i : X \to V$. Then we define the topological index to be

$$\mathrm{ind}_t : KU_{G,c}(TX) \xrightarrow{i_!} KU_{G,c}(TV) \simeq R(G)$$

The index theorem can be stated as follows.

Theorem

For a compact G-manifold X, let P be a G-ellptic operator over G-vector bundles E, F over X. Then we have

$$\operatorname{ind}_t P \simeq \operatorname{ind}_a P$$

This proof is identical to the non-equivariant case, that is, we check that ind_a commutes with open embeddings and Thom isomorphisms. We skip the proof, since there is nothing new.

The Atiyah-Segal completion theorem

Finally we present some other aspects of equivariant K-theory. We will not present any details here.

Definition

We recall that a spectrum X is a sequence of spaces X_n such that for each n we have $\Omega X_n = X_{n+1}$. They represent the cohomology theories. We introduce a likewise construction involving G.
Consider a finite dimensional G-representation V, and let $\mathbb{S}^V$ be the one-point compactification of V, which is a representation sphere of G.
Now we consider the set of irreducible representations of G, say $U_{\alpha\alpha\in A}$.
Then consider $\mathcal{U} = \bigoplus U_\alpha^{\oplus\infty}$. We let a genuine G-spectra X to be a set of pointed G-spaces indexed by finite dimensional subrepresentations of $\mathcal{U}$, say X_V. Moreover, these spaces satisfy for $V \subseteq W$,
$X_V \simeq \mathrm{Hom}_G(\mathbb{S}^{W-V}, X_W)$.

Construction

For a pointed G-space X, we may form the G-suspension spectrum $\Sigma^\infty X$ defined as follows. Firstly we let $QX = \varinjlim_V \Omega^V \Sigma^V X$, where this colimit is taken over all finite dimensional subrepresentations of $\mathcal{U}$. Then we let $(\Sigma^\infty X)_W = Q\mathbb{S}^W X$.

We define the equivariant cohomology represented by a G-spectrum X by sending a G-space Y to the Abelian group $\pi_0 \operatorname{Hom}(\Sigma^\infty Y, X)$.

For a generalized cohomology theory representable by a spectrum E, one may form the corresponding Borel equivariant cohomology theory X^{hG} by assigning $Y \mapsto E^*(Y/G)$. Here the quotient is the homotopy quotient.

Theorem

The previous construction KU_G is a cohomology theory representable by a spectrum. Moreover, letting KU denote the spectrum representing the usual complex K-theory, we have a natural map $KU_G \to KU^{hG}$ such that

$$R(G) \simeq KU_G(*) \to KU^G(*) \simeq KU(BG)$$

is the completion at the augmentation ideal, which is the kernel of $R(G) \to \mathbb{Z}$ by sending a representation to its rank.

This indicates the importance of studying equivariant K-theory, and furthermore, the importance of studying G-spectra.

Thank you!

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